Classical Mechanics (1)

Axiom1.1 (Newtonian Formalism)

Suppose there are N particles, such that their masses, position at time t_0 ,velocity at time t_0 are given by $m_i, \mathbf{r}_i |_{t=t_0}, \dot{\mathbf{r}}_i |_{t=t_0}$ for *i=1,...N* respectively. Suppose the force between them are given by $\mathbf{F}_{ij} = \mathbf{F}_{ij}(\mathbf{r}_i - \mathbf{r}_j)$ for $\forall i, j = 1,\ldots,N$, such that they satisfies

$$\begin{cases} \mathbf{F}_{ij} = \mathbf{F}_{ji} \\ (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = 0 \end{cases} \qquad \text{.....(1.1.1) and (1.1.2)}$$

Then $\{\mathbf{r}_i(t)\}_{i=1}^{N}$ for all time t can be determined by

$$m_i \ddot{\mathbf{r}}_{ij}(t) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{F}_{ij} \qquad \text{.....(1.1.3)}$$

Def 1.2 (Definition of Rigid Body)

Particles *i=1,...,N* form a rigid body iff $\exists \Delta r_{ij}$ for *i=1,...N*; *j=1,2,3* such that $\sum_{i=1}^{N} \Delta r_{ij} = 0$ for *j=1,2,3* and for any instance, there exist $\phi, \theta, \psi, \mathbf{R}$ such that the positions of those particles can be given by

$$\mathbf{r}_i = \mathbf{R}(t) + \Delta \mathbf{r}_i(t) \qquad \text{.......(1.2.1)}$$

where

$$\Delta \mathbf{r}_i(t) = \sum_{j=1}^{3} \Delta r_{ij} \mathbf{e}_j(t) \qquad \text{.......(1.2.2)}$$

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{bmatrix} \qquad \text{.......(1.2.3)}$$

Def 1.3 (Angular Momentum)

The total angular momentum of $i=1,\ldots,N$ particles w.r.t. a fixed point **r** is given by

$$\mathbf{L_r} = \sum_{i=1}^{N} m_i (\mathbf{r}_i(t) - \mathbf{r}) \times \dot{\mathbf{r}}_i(t) \qquad \ldots\ldots..(1.3.1)$$

<u>Thm 1.4 (Conservation of Angular Momentum)</u>

Suppose there are $i=1,\ldots,N$ particles which are isolated from the rest of the world. Then for any fixed point **r**,

$$\dot{\mathbf{L}}_\mathbf{r} = 0$$

Proof:

By application of (1.1.1) and (1.1.2). Detail refer to "Snow Mountain Book" 9-10-94

<u>Thm 1.5</u>

Suppose particles $i=1,\ldots,N$ formed a rigid body. Let **r** be a fixed point. Then the relation of the total angular momentum of these N particles w.r.t. point **r** and **R** are related by:

$$\mathbf{L_r} = (\mathbf{R} - \mathbf{r}) \times M\dot{\mathbf{R}} + \mathbf{L_R} \qquad \ldots\ldots(1.5.1)$$

where $$M = \sum_{i=1}^{N} m_i \qquad \ldots\ldots(1.5.2)$$

R can be referred to Def 1.2.

<u>Thm 1.6</u>

Suppose particles $i=1,\ldots,N$ form a rigid body. Define

$$P_{ij} = \sum_{i=1}^{N} m_i \Delta r_{ij} \Delta r_{ik} \quad \text{for } j,k=1,2,3 \qquad \ldots\ldots.(1.5.3)$$

Define $$\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} = \begin{bmatrix} P_{22}+P_{33} & -P_{21} & -P_{31} \\ -P_{12} & P_{11}+P_{33} & -P_{32} \\ -P_{13} & -P_{23} & P_{11}+P_{22} \end{bmatrix} \qquad \ldots\ldots.(1.5.4)$$

Define $$\boldsymbol{\omega} = \frac{1}{2}\sum_{j=1}^{3} \mathbf{e}_j \times \dot{\mathbf{e}}_j$$ ……..(1.5.5)

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ can be referred to Def 1.2.

Write $$\boldsymbol{\omega} = \sum_{j=1}^{3} \omega_j \mathbf{e}_j$$ ……..(1.5.6)

and $$\mathbf{L}_{\mathbf{R}} = \sum_{j=1}^{3} L_j \mathbf{e}_j$$ …….(1.5.7)

then $$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$ ……..(1.5.8)

On the other hand, it can be proven that

$\Delta\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \Delta\mathbf{r}_i$ for $\forall i$, ……(1.5.9)

where we recall here $\Delta\mathbf{r}_i$ is defined in (1.2.2).

<u>Thm 1.7</u>

Write $$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$ …..(1.7.1)

Since **P** is self-adjoint, $\exists$**A**, a 3×3 unitary matrix such that $\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}$ is diagonal. Define $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ by

$$\begin{bmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$ …….(1.7.2)

and $$\mathbf{P}' = \begin{bmatrix} P'_{11} & P'_{12} & P'_{13} \\ P'_{21} & P'_{22} & P'_{23} \\ P'_{31} & P'_{32} & P'_{33} \end{bmatrix} \qquad \text{…….(1.7.3)}$$

where $$P'_{jk} = \sum_{i=1}^{N} m_i \Delta r'_{ij} \Delta r'_{ik} \qquad \text{……..(1.7.4)}$$

$$\sum_{j=1}^{3} \Delta r'_{ij} \mathbf{e}'_j = \sum_{j=1}^{3} \Delta r_{ij} \mathbf{e}_j \qquad \text{……..(1.7.5)}$$

Then $\mathbf{P}' = \mathbf{A}^{\mathrm{T}} \mathbf{P} \mathbf{A}$ ……..(1.7.6)

Proof:

Since $$\begin{bmatrix} \Delta r'_{i1} & \Delta r'_{i2} & \Delta r'_{i3} \end{bmatrix} \begin{bmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{bmatrix} = \begin{bmatrix} \Delta r_{i1} & \Delta r_{i2} & \Delta r_{i3} \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \Delta r_{i1} & \Delta r_{i2} & \Delta r_{i3} \end{bmatrix} \mathbf{A} \begin{bmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{bmatrix},$$

$$\Rightarrow \begin{bmatrix} \Delta r'_{i1} & \Delta r'_{i2} & \Delta r'_{i3} \end{bmatrix} = \begin{bmatrix} \Delta r_{i1} & \Delta r_{i2} & \Delta r_{i3} \end{bmatrix} \mathbf{A}$$

$$\Rightarrow \sum_{i=1}^{3} m_i \begin{bmatrix} \Delta r'_{i1} \\ \Delta r'_{i2} \\ \Delta r'_{i3} \end{bmatrix} \begin{bmatrix} \Delta r'_{i1} & \Delta r'_{i2} & \Delta r'_{i3} \end{bmatrix} = \sum_{i=1}^{3} m_i \mathbf{A}^{\mathrm{T}} \begin{bmatrix} \Delta r_{i1} \\ \Delta r_{i2} \\ \Delta r_{i3} \end{bmatrix} \begin{bmatrix} \Delta r_{i1} & \Delta r_{i2} & \Delta r_{i3} \end{bmatrix} \mathbf{A}$$

$$\Rightarrow \mathbf{P} = \mathbf{A}^{\mathrm{T}} \mathbf{P} \mathbf{A}$$

<u>Thm 1.8</u>

Recall particles i=1,…,N form a rigid body. Let for every particle i, there is a external force $\mathbf{F}_i^{\text{ext}}$ acting on it. Then

$$M\ddot{\mathbf{R}} = \sum_{i=1}^{N} \mathbf{F}_i^{\text{ext}} \qquad \text{…..(1.8.1)}$$

Write $$\mathbf{I} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \qquad \text{……..(1.8.2)}$$

Suppose **I** can be given by $I = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}$ …….(1.8.3)

Define $\mathbf{\Gamma} = \sum_{i=1}^{N} \Delta \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}$ …….(1.8.4)

Write $\mathbf{\Gamma} = \sum_{j=1}^{3} \Gamma_j \mathbf{e}_j$ ……..(1.8.5)

Then we have the Euler's equation of motion:

$$\begin{cases} I_{11}\dot{\omega}_1 + \omega_2\omega_3(I_{33} - I_{22}) = \Gamma_1 \\ I_{22}\dot{\omega}_2 + \omega_1\omega_3(I_{11} - I_{33}) = \Gamma_2 \\ I_{33}\dot{\omega}_3 + \omega_1\omega_2(I_{22} - I_{11}) = \Gamma_3 \end{cases} \quad \text{……..(1.8.6)}$$

Proof: Refer to Snow Mountain Book Classcial Mechanics 27-12-94.

<u>Def 1.9</u>

The total kinetic energy of particles *i=1,…,N* is defined by

$$\text{K.E.} = \sum_{i=1}^{N} \frac{1}{2} m_i \, |\dot{\mathbf{r}}_i(t)|^2 \quad \text{……..(1.9.1)}$$

<u>Thm 1.10</u>

Suppose particles *i=1,…,N* form a rigid body. Then the total kinetic energy of these particles can be given by

$$(\text{K.E.})_{\text{total}} = (\text{K.E.})_{\text{translational}} + (\text{K.E.})_{\text{rotational}} \quad \text{……(1.10.1)}$$

where $(\text{K.E.})_{\text{translational}} = \frac{1}{2} M \, |\dot{\mathbf{R}}|^2$ …….(1.10.2)

$$(\text{K.E.})_{\text{rotational}} = \frac{1}{2}(I_{11}\omega_1^2 + I_{22}\omega_2^2 + I_{33}\omega_3^2) + I_{12}\omega_1\omega_2 + I_{13}\omega_1\omega_3 + I_{23}\omega_2\omega_3 \quad \text{…..(1.10.3)}$$

Definition **R**, *M* should be referred to Def 1.2; $\omega_1, \omega_2, \omega_3$ should be referred to (1.5.5); I_{ij} for *i,j=1,2,3* should be referred to (1.5.4).

Thm 1.11 (Lagrangian Formulation)

With reference to Axiom 1.1, suppose $\exists$ $U_{ij}(r)$ for $i,j=1,\ldots,N$ such that $U_{ij}(r)=U_{ji}(r)$ for $\forall$ *i,j* and $\mathbf{F}ij$ can be expressed as

$$\mathbf{F}_{ij} = -\frac{\partial U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)}{\partial \mathbf{r}_i} \qquad \ldots\ldots..(1.11.1)$$

for $\forall$ *i,j*. Then define the total potential energy:

$$V = \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N} U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) \qquad \ldots\ldots..(1.11.2)$$

Let us write the total kinetic energy defined in Def 1.9 as *T*, i.e.,

$$T = \sum_{i=1}^{N}\frac{1}{2}m_i|\dot{\mathbf{r}}_i|^2 \qquad \ldots\ldots..(1.11.3)$$

Suppose now there exist a set of coordinates, $q_1,\ldots,q_{N_q}$, where $N_q \le 3N$ such that the positions of those particles at any time can be expressed in terms of these coordinates, i.e. $\mathbf{r}_i = \mathbf{r}_i(q_1,\ldots,q_{N_q})$ for $i=1,\ldots,N$. Define the Lagrangian of the system:

$$L = T - V \qquad \ldots\ldots..(1.11.4)$$

Suppose now these $q_1,\ldots,q_{N_q}$ coordinates are subjected to *p* constraint ($p \le N_q$), i.e.

$$f_k(q_1,\ldots,q_p,t) = 0 \qquad \text{for } k=1,\ldots,p \qquad \ldots\ldots.(1.11.5)$$

Then it can be proven that $\exists \lambda_k(t)$ for $k=1,\ldots,p$ such that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \sum_{k=1}^{p}\lambda_k(t)\frac{\partial f_k}{\partial q_j} = 0 \qquad \text{for } j=1,\ldots,N_q \qquad \ldots\ldots\ldots.(1.11.6)$$

Thm 1.16 (Hamiltonian Formulation)

Define canonical momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$, for $i=1,\dots,N$ ……(1.16.1)

Define Hamiltonian function

$$H(p_1,\dots,p_N,q_1,\dots,q_N,t) = \{\sum_{i=1}^{N} \dot{q}_i p_i\} - L$$ ……(1.16.2)

By expand *dH* in R.H.S.:

$$\begin{aligned} dH &= \{\sum_{i=1}^{N} \dot{q}_i dp_i + p_i d\dot{q}_i\} - \{\sum_{i=1}^{N} \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i\} - \frac{\partial L}{\partial t} dt \\ &= \sum_{i=1}^{N} \{\dot{q}_i dp_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i\} - \frac{\partial L}{\partial t} dt \\ &= \sum_{i=1}^{N} \{\dot{q}_i dp_i - \dot{p}_i dq_i\} - \frac{\partial L}{\partial t} dt \end{aligned}$$

and in L.H.S.:

$$dH = \sum_{i=1}^{N} \{\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i\} + \frac{\partial H}{\partial t} dt$$

and compare the two side, we have

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$ and $$\dot{q}_i = \frac{\partial H}{\partial p_i}$$, for $i=1,\dots,N$ …….(1.16.3)