Electro-Magnetism (2)

Thm 2.1

$$\rho(\mathbf{r},t)=\frac{1}{\Delta x\Delta y\Delta z}\sum_{\substack{i:\\ x\le x_i(t)<x+\Delta x\\ y\le y_i(t)<y+\Delta y\\ z\le z_i(t)<z+\Delta z}} q_i \qquad \text{.......(2.1.1)}$$

$$\mathbf{J}(\mathbf{r},t)=\frac{1}{\Delta x\Delta y\Delta z}\sum_{\substack{i:\\ x\le x_i(t)<x+\Delta x\\ y\le y_i(t)<y+\Delta y\\ z\le z_i(t)<z+\Delta z}} q_i\dot{\mathbf{r}}_i(t) \qquad \text{.......(2.1.2)}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_o}{4\pi}\int_{\substack{\text{all}\\ \text{space}}}\frac{\mathbf{J}(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r}-\mathbf{r}'|}d^3r' \qquad \text{......(2.1.3)}$$

$$\varphi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_o}\int_{\substack{\text{all}\\ \text{space}}}\frac{\rho(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r}-\mathbf{r}'|}d^3r' \qquad \text{......(2.1.4)}$$

$$\mathbf{E}(\mathbf{r},t)=-\nabla\varphi(\mathbf{r},t)-\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) \qquad \text{......(2.1.5)}$$

$$\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}(\mathbf{r},t) \qquad \text{.......(2.1.6)}$$

$$\frac{d}{dt}\left[\frac{m_i\dot{\mathbf{r}}_i(t)}{\sqrt{1-\frac{1}{c^2}|\dot{\mathbf{r}}_i(t)|^2}}\right]=q_i[E(\mathbf{r}_i(t),t)+\dot{\mathbf{r}}_i(t)\times\mathbf{B}(\mathbf{r}_i(t),t)] \qquad \text{......(2.1.7)}$$

Thm 2.2

The $\mathbf{A}(\mathbf{r},t)$, $\varphi(\mathbf{r},t)$ given in (2.1.3), (2.1.4) can be proven to satisfy the following three equation:

$$\nabla^2\varphi(\mathbf{r},t)-\frac{1}{c^2}\frac{\partial^2\varphi(\mathbf{r},t)}{\partial t^2}=-\frac{\rho(\mathbf{r},t)}{\varepsilon_o} \qquad \text{......(2.2.1)}$$

$$\nabla^2 \mathbf{A}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}(\mathbf{r},t)}{\partial t^2} = -\mu_o \mathbf{J}(\mathbf{r},t) \qquad \ldots\ldots(2.2.2)$$

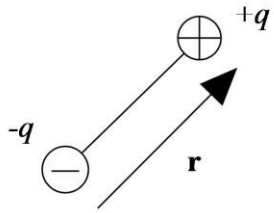
$$\nabla \cdot \mathbf{A}(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial \varphi(\mathbf{r},t)}{\partial t} = 0 \qquad \ldots\ldots.(2.2.3)$$

<u>Thm 2.3</u>

From (2.2.1), (2.2.2), (2.2.3), we can derive the Maxwell equation in vaccum

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = \frac{\rho(\mathbf{r},t)}{\varepsilon_o} \qquad \ldots\ldots.(2.3.1)$$
$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} \qquad \ldots\ldots..(2.3.2)$$
$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0 \qquad \ldots\ldots\ldots(2.3.3)$$
$$\nabla \times \mathbf{B}(\mathbf{r},t) = \mu_o \mathbf{J}(\mathbf{r},t) + \mu_o\varepsilon_o \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} \qquad \ldots\ldots..(2.3.4)$$

<u>Def 2.4</u>

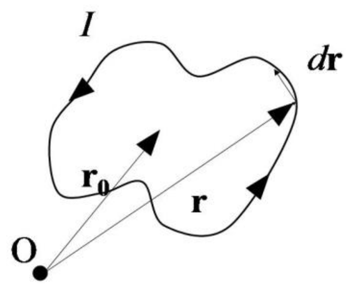


The dipole moment of the dipole as shown in the figure is given by

$$\mathbf{p} = q\mathbf{r} \qquad \ldots..(2.4.1)$$

The polarization **P** at point **r** is given by

$$\mathbf{P}(\mathbf{r},t) = \frac{1}{\Delta x \Delta y \Delta z} \sum_{\substack{i: \\ x \le x_i(t) < x+\Delta x \\ y \le y_i(t) < y+\Delta y \\ z \le z_i(t) < z+\Delta z}} \mathbf{p}_i \qquad \ldots\ldots(2.4.2)$$



The magnetic dipole moment of the magnetic dipole given in the figure is given by

$$\mathbf{m} = \frac{1}{2} I \oint (\mathbf{r} - \mathbf{r}_0) \times d\mathbf{r} \qquad \ldots\ldots(2.4.3)$$

The magnetization M of point r is given by

$$\mathbf{M}(\mathbf{r}, t) = \frac{1}{\Delta x \Delta y \Delta z} \sum_{\substack{i: \\ x \le x_i(t) < x + \Delta x \\ y \le y_i(t) < y + \Delta y \\ z \le z_i(t) < z + \Delta z}} \mathbf{m}_i$$

Thm 2.5

Let the space is filled with medium, then the space will be filled with electric and magnetic dipole. The divergence of **P** will give a effect of charge, named bound charge, which is given by

$$\rho_b = -\nabla \cdot \mathbf{P} \qquad \ldots\ldots(2.5.1)$$

The curl of **M** will give an effect of current, named bound current, which is given by

$$\mathbf{J}_b = \nabla \times \mathbf{M} \qquad \ldots\ldots.(2.5.2)$$

Also, the derivative of **P** w.r.t. time will give an effect of current, named polarization current, which is given by

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} \qquad \ldots\ldots(2.5.3)$$

$\therefore$ The Maxwell equation should be modified to

$$\nabla\cdot\mathbf{E}=\frac{1}{\varepsilon_o}(\rho_f+\rho_b)=\frac{1}{\varepsilon_o}(\rho_f-\nabla\cdot\mathbf{P}) \qquad \ldots\ldots(2.5.4)$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \qquad \ldots\ldots.(2.5.5)$$

$$\nabla\cdot\mathbf{B}=0 \qquad \ldots..(2.5.6)$$

$$\begin{aligned}\nabla\times\mathbf{B}&=\mu_o(\mathbf{J}_f+\mathbf{J}_b+\mathbf{J}_p)+\mu_o\varepsilon_o\frac{\partial\mathbf{E}}{\partial t}\\&=\mu_o(\mathbf{J}_f+\nabla\times\mathbf{M}+\frac{\partial\mathbf{P}}{\partial t})+\mu_o\varepsilon_o\frac{\partial\mathbf{E}}{\partial t}\end{aligned} \qquad \ldots\ldots(2.5.7)$$

Define

$$\mathbf{D}=\varepsilon_o\mathbf{E}+\mathbf{P} \qquad \ldots\ldots.(2.5.8)$$

$$\mathbf{H}=\frac{1}{\mu_o}\mathbf{B}-\mathbf{M} \qquad \ldots\ldots.(2.5.9)$$

Then eqt (2.5.4) to (2.5.7) can be rewritten as

$$\nabla\cdot\mathbf{D}=\rho_f \qquad \ldots\ldots.(2.5.10)$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \qquad \ldots\ldots.(2.5.11)$$

$$\nabla\cdot\mathbf{B}=0 \qquad \ldots\ldots.(2.5.12)$$

$$\nabla\times\mathbf{H}=\mathbf{J}_f+\frac{\partial\mathbf{D}}{\partial t} \qquad \ldots\ldots.(2.5.13)$$

<u>Thm 2.6</u>

In linear medium, there exist a constant χ_e (electric susceptibility) and χ_m (magnetic susceptibility), which its value depends on what the medium is and such that

$$\begin{cases}\mathbf{P}=\varepsilon_o\chi_e\mathbf{E}\\ \mathbf{M}=\chi_m\mathbf{H}\end{cases} \qquad \ldots\ldots(2.6.1)\text{ and }(2.6.2)$$

From this two equation, we can derive that there exist two constant ε (permittivity of the medium) and μ (permeability of the medium) such that

$$\mathbf{D}=\varepsilon\mathbf{E} \qquad \ldots\ldots.(2.6.3)$$

$$\mathbf{H}=\frac{1}{\mu}\mathbf{B} \qquad \ldots\ldots.(2.6.4)$$

Also dielectric constant κ is defined as $\kappa=\varepsilon/\varepsilon_o$ …….(2.6.5)

Thm 2.7

Let U_1, U_2 are the solution to V: $\nabla^2 V = -4\pi\rho$ in Ω. Let U_1=U_2 on $\partial\Omega$ (Dirrichlet cond't) or $\nabla U_1 \cdot \mathbf{n} = \nabla U_2 \cdot \mathbf{n}$ on $\partial\Omega$ (Neumann cond't).

$\Rightarrow U_1 - U_2 = \text{const in } \Omega$

Proof: Refer to AEM handwritten notes Thm 3.

Thm 2.8

Let Ω be a closed volume in space. Let $\nabla^2\Phi = -4\pi\rho$ in Ω and $\Phi = \Phi_1$ on $\partial\Omega$ (Dirrchelet cond't). Let $G(\mathbf{r}_0,\mathbf{r}) = \dfrac{1}{|\mathbf{r}_0 - \mathbf{r}|} + F(\mathbf{r}_0,\mathbf{r})$, where $\mathbf{r}_0 \in \Omega$. $G(\mathbf{r}_0,\mathbf{r})|_{\mathbf{r}\in\partial\Omega} = 0$,
$\nabla^2 F(\mathbf{r}_0,\mathbf{r}) = 0$ for $\mathbf{r} \in \Omega$.

$$\Rightarrow \Phi(\mathbf{r}_0) = \int_\Omega G(\mathbf{r}_0,\mathbf{r})\rho(\mathbf{r})d^3r - \frac{1}{4\pi}\oint_{\partial\Omega}\Phi_1(\mathbf{r})\nabla G(\mathbf{r}_0,\mathbf{r})\cdot d\mathbf{a}\ (\mathbf{a} \text{ is pointing away from } \Omega)$$

for $\forall \mathbf{r}_0 \in \Omega$ ……..(2.8.1)

Proof: Refer to AEM handwritten notes Thm 4.

Thm 2.9

Let Ω be a closed volume in space. Let $\nabla^2\Phi = -4\pi\rho$ in Ω and $\nabla\Phi\cdot\mathbf{n} = g$ ($\mathbf{n}$ is pointing away from Ω) on $\partial\Omega$. Let $G(\mathbf{r}_0,\mathbf{r}) = \dfrac{1}{|\mathbf{r}_0 - \mathbf{r}|} + F(\mathbf{r}_0,\mathbf{r})$, where $\mathbf{r}_0 \in \Omega$,
$\nabla G(\mathbf{r}_0,\mathbf{r})\cdot\mathbf{n}|_{\mathbf{r}\in\partial\Omega} = -\dfrac{4\pi}{\text{Area}(\partial\Omega)}$, $\nabla^2 F(\mathbf{r}_0,\mathbf{r}) = 0$ for $\mathbf{r} \in \Omega$

$$\Rightarrow \Phi(\mathbf{r}_0) = \int_\Omega G(\mathbf{r}_0,\mathbf{r})\rho(\mathbf{r})d^3r + \frac{1}{4\pi}\int_{\partial\Omega} G(\mathbf{r}_0,\mathbf{r})g(\mathbf{r})d\mathbf{a} + \frac{1}{\text{Area}(\partial\Omega)}\int_{\partial\Omega}\Phi(\mathbf{r})d\mathbf{a}$$

……(2.9.1)

Proof: Refer to AEM handwritten notes Thm 4.

Lemma 2.10

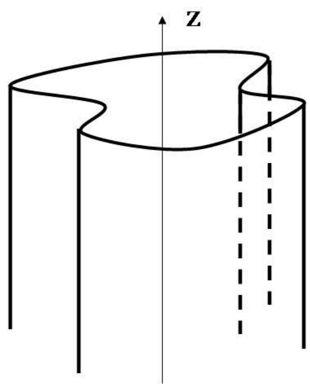
In order to make Thm 2.9 useful, use a Ω such that $\text{Area}(\partial\Omega) \to +\infty$.

Thm 2.11

Let S be a sphere of radius R, centered at the origin. Let $s>R$, then

$$\frac{1}{|\mathbf{r}-s\mathbf{e}_r|}+\frac{-\frac{R}{s}}{|\mathbf{r}-\frac{R^2}{s}\mathbf{e}_r|}\Big|_{\mathbf{r}\in\partial S}\equiv 0 \qquad \text{.....(2.11.1)}$$

Thm 2.12



Let Ω be a cylinder in space, such that its axis is parallel to z-axis and its cross-section shape can be arbitary. Let electric charge distribution ρ inside Ω is independent of z, i.e. $\rho=\rho(x,y)$. From electrostatic, we have the law $\nabla^2\Phi=-4\pi\rho$ in Ω. Suppose $\Phi=\Phi_1$ on $\partial\Omega$, where $\Phi_1=\Phi_1(x,y)$, then we can assume $\Phi=\Phi(x,y)$. Let $G(\mathbf{r}_0,\mathbf{r})=\ln|\mathbf{r}_0-\mathbf{r}|+F(\mathbf{r}_0,\mathbf{r})$, where $\mathbf{r}_0=x_0\mathbf{e}_x+y_0\mathbf{e}_y, \mathbf{r}=x\mathbf{e}_x+y_y, \mathbf{r}_0\in\Omega$.

$G(\mathbf{r}_0,\mathbf{r})|_{\mathbf{r}\in\partial\Omega}=0$,

$\nabla^2F(\mathbf{r}_0,\mathbf{r})=0 \text{ for } \mathbf{r}\in\Omega$

$$\Rightarrow \Phi(\mathbf{r}_0)=\int_\Omega G(\mathbf{r}_0,\mathbf{r})\rho(\mathbf{r})d^2r-\frac{1}{4\pi}\oint_{\partial\Omega}\Phi_1(\mathbf{r})\nabla G(\mathbf{r}_0,\mathbf{r})\cdot d\mathbf{l}_\perp \quad \text{for } \forall\mathbf{r}_0\in\Omega$$

where $\mathbf{l}_\perp$ is pointing away from Ω.

Thm 2.13

Let C be a circle of radius R on a plane, centered at the origin. Let $s>R$, then

$$\ln|\mathbf{r}-s\mathbf{e}_\rho|-\ln|\mathbf{r}-\frac{R^2}{s}\mathbf{e}_\rho|\Big|_{\mathbf{r}\in\partial C}\equiv \text{a constant},$$

where $\mathbf{r},\mathbf{e}_\rho$ lies on the aforementioned plane.

Axiom 2.13a

We are living in a four dimensional continuum, such that every point in this continuum can be described by four co-ordinates: (x_0,x_1,x_2,x_3)

Axiom 2.14

Let $T^{\alpha\cdots}_{\beta\gamma\cdots}$ be a tensor in the coordinate system (x_0,x_1,x_2,x_3). Let $T'^{\alpha\cdots}_{\beta\gamma\cdots}$ be the same tensor in the coordinate system (x'_0,x'_1,x'_2,x'_3), then $T^{\alpha\cdots}_{\beta\gamma\cdots}, T'^{\alpha\cdots}_{\beta\gamma\cdots}$ are related by

$$T'^{\mu\cdots}_{\nu\sigma\cdots}=\frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x_\beta}{\partial x'_\nu}\frac{\partial x_\gamma}{\partial x'_\sigma}\cdots T^{\alpha\cdots}_{\beta\gamma\cdots} \qquad \text{.......(2.14.1)}$$

Axiom 2.15

In every point of our four dimensional continuum, we can define the covariant metric tensor $g_{\mu\nu}$. Then the contravariant metric tensor is found by

$$g_{\mu\nu}g^{\nu\sigma}=\delta_{\mu\sigma} \qquad \text{.......(2.15.1)}$$

Then for any tensor at that point, we have

$$T^{\alpha\cdots}_{\beta\gamma\cdots}g_{\alpha\delta}=T^{\cdots}_{\delta\beta\gamma\cdots} \qquad \text{........(2.15.2)}$$

and

$$T^{\alpha\cdots}_{\beta\gamma\cdots}g^{\beta\delta}=T^{\beta\alpha\cdots}_{\gamma\cdots} \qquad \text{......(2.15.3)}$$

Thm 2.16

Let Ω be a region in our four dimensional continuum which is enough small. Then we can always perform coordinate transformation (x_0,x_1,x_2,x_3) to (x'_0,x'_1,x'_2,x'_3), such that in the region of Ω in this new coordinate system, $x'_0=ct, x'_1=x, x'_2=y, x'_3=z$ and that the covariant metric tensor is given by

$$g_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \qquad \ldots\ldots.(2.16.1)$$

Thm 2.17

In the region Ω in the new coordinate system described in the last theorem, we can talk about all the electromagnetism we mentioned before. We propose that from many of the physical quantities in electromagnetism, we can construct tensors:

Current density four vector:

$$J^{\mu} = (c\rho, J_x, J_y, J_z) \qquad \ldots\ldots.(2.17.1)$$

Four vector potential

$$A^{\mu} = (\frac{1}{c}\varphi, A_x, A_y, A_z) \qquad \ldots\ldots..(2.17.2)$$

Field tensor

$$F^{\mu\nu} = \begin{bmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ -\frac{1}{c}E_x & 0 & B_z & -B_y \\ -\frac{1}{c}E_y & -B_z & 0 & B_x \\ -\frac{1}{c}E_z & B_y & -B_x & 0 \end{bmatrix} \qquad \ldots\ldots..(2.17.3)$$

Recall in Thm 2.1 that the position of a particle is described as

$$\mathbf{r}_i(t) = x_i(t)\mathbf{e}_x + y_i(t)\mathbf{e}_y + z_i(t)\mathbf{e}_z \qquad \text{…….(2.17.4)}$$

Define the Four velocity of this particle

$$v_i^\mu = (\gamma, \frac{\gamma}{c}\dot{x}_i(t), \frac{\gamma}{c}\dot{y}_i(t), \frac{\gamma}{c}\dot{z}_i(t)) \qquad \text{……(2.17.5)}$$

where $\gamma = \left(1 - \frac{|\dot{\mathbf{r}}_i(t)|^2}{c^2}\right)^{-1/2}$ ……..(2.17.6)

Def 2.18

A path in the four dimensional continuum can be described as

$$x_\sigma = x_\sigma(\lambda) \qquad \text{……..(2.18.1)}$$

for σ=0,1,2,3. Suppose along this path, we can define a tensor $A^{\alpha\cdots}_{\beta\gamma\cdots}(\lambda)$, then the covariant derivative of this tensor at a point on the path is given by

$$\frac{DA^{\alpha\cdots}_{\beta\gamma\cdots}}{D\lambda} = \frac{dA^{\alpha\cdots}_{\beta\gamma\cdots}}{d\lambda} - \frac{\delta A^{\alpha\cdots}_{\beta\gamma\cdots}}{d\lambda} \qquad \text{…….(2.18.2)}$$

where $\delta A^{\alpha\cdots}_{\beta\gamma\cdots}$ can be generated using the formula:

$$\begin{cases} \delta q^\mu = -\Gamma^\mu_{\sigma\rho} q^\sigma dx^\rho \\ \delta q_\mu = +\Gamma^\nu_{\mu\rho} q_\nu dx^\rho \\ \delta(A^{\alpha\cdots}_{\beta\gamma\cdots} q_\alpha q^\beta q^\gamma) = 0 \end{cases} \qquad \text{……(2.18.3a, b and c)}$$

where $dx^\rho = (x_\rho(\lambda + d\lambda) - x_\rho(\lambda))$ ……..(2.18.4)

Def 2.19

Let $A^{\alpha\cdots}_{\beta\gamma\cdots}$ be a tensor which is defined in every point in the four dimensional continuum, then we write

$$A^{\alpha\cdots}_{\beta\gamma\cdots;\tau} = \frac{DA^{\alpha\cdots}_{\beta\gamma\cdots}}{D\lambda} \qquad \text{…….(2.19.1)}$$

where we have assume a path given by

$$x_\sigma(\lambda) = \begin{cases} \lambda & \text{for } \sigma = \tau \\ x_{\sigma 0} & \text{for } \sigma \neq \tau \end{cases} \quad \text{……..(2.19.2)}$$

Thm 2.20

The tensors defined in Thm 2.17 have the following relations to each other

$$F^\mu{}_\nu = A^\mu{}_{;\nu} - A^\nu{}_{;\mu} \quad \text{……..(2.20.1)}$$

$$F^{\mu\nu}{}_{;\nu} = \mu_o J^\mu \quad \text{…….(2.20.2)}$$

$$m\frac{Dv_i^\mu}{Ds} = -\frac{q}{c} v_{i\nu} F^{\mu\nu} \quad \text{…….(2.20.3)}$$

In the covariant derivatives of v^μ w.r.t. *s* in (2.20.3), we have assumed v^μ is defined along the world line of the particle concerned, where the world line of the particle is given by

$$x_\sigma = x_\sigma(s) \quad \text{…….(2.20.4)}$$

such that for any *s*, we have

$$\sum_{\mu,\nu=0}^{3} g_{\mu\nu} \frac{dx_\mu(s)}{ds}\frac{dx_\nu(s)}{ds} = 1 \quad \text{…….(2.20.5)}$$

(For the details about covariant differentiation, refer to “General Relativity”, by I. R. Kenyon)

Thm 2.21

Suppose we are in Ω and the new coordinate system described in Thm 2.16. Suppose now in Ω, there are many particles, such that their positions are described by

$$\mathbf{r}_i(t) = \sum_{\alpha=x,y,z} r_{i\alpha}(t)\mathbf{e}_\alpha \quad \text{……(2.21.1)}$$

Define the energy-momentum tensor of these particles:

$$T^{\alpha 0}(\mathbf{r},t) = \frac{1}{\Delta x \Delta y \Delta z} \sum_{\substack{i: \\ x \le r_{ix}(t) < x+\Delta x \\ y \le r_{iy}(t) < y+\Delta y \\ z \le r_{iz}(t) < z+\Delta z}} c\gamma_i m_i \dot{r}_{i\alpha}(t) \qquad \text{for } \alpha = 1,2,3 \quad \text{……(2.21.2)}$$

$$T^{00}(\mathbf{r},t)=\frac{1}{\Delta x\Delta y\Delta z}\sum_{\substack{i:\\ x\le r_{ix}(t)<x+\Delta x\\ y\le r_{iy}(t)<y+\Delta y\\ z\le r_{iz}(t)<z+\Delta z}}\gamma_i m_i c^2 \qquad \text{.......(2.21.3)}$$

$$T^{\alpha\beta}(\mathbf{r},t)=\frac{1}{\Delta x\Delta y\Delta z}\sum_{\substack{i:\\ x\le r_{ix}(t)<x+\Delta x\\ y\le r_{iy}(t)<y+\Delta y\\ z\le r_{iz}(t)<z+\Delta z}}\gamma_i m_i \dot{r}_{i\alpha}(t)\dot{r}_{i\beta}(t) \qquad \text{for } \alpha,\beta=1,2,3 \qquad \text{.....(2.21.4)}$$

$$T^{0\beta}(\mathbf{r},t)=\frac{1}{\Delta x\Delta y\Delta z}\sum_{\substack{i:\\ x\le r_{ix}(t)<x+\Delta x\\ y\le r_{iy}(t)<y+\Delta y\\ z\le r_{iz}(t)<z+\Delta z}}\gamma_i m_i c\dot{r}_{i\beta}(t) \qquad \text{for } \beta=1,2,3 \qquad \text{.......(2.21.5)}$$

where $\gamma_i=\left(1-\frac{|\dot{\mathbf{r}}_i(t)|^2}{c^2}\right)^{-1/2}$ ....(2.21.6)

Define the energy-momentum tensor for the electromagnetic field

$$T_{\mathrm{EM}}^{\alpha\beta}=\frac{1}{\mu_o}[-F^{\alpha}{}_{\gamma}F^{\beta\gamma}+\frac{1}{4}g^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}] \qquad \text{.......(2.21.7)}$$

Then we have

$$(T^{\alpha\beta}+T_{\mathrm{EM}}^{\alpha\beta})_{;\beta}=0 \qquad \text{.......(2.21.8)}$$

<u>Thm 2.22</u>

Suppose we are in Ω and the new coordinate system described in Thm 2.16,

$$T_{\mathrm{EM}}^{00}=\frac{\varepsilon_o}{2}|\mathbf{E}|^2+\frac{1}{2\mu_o}|\mathbf{B}|^2 \qquad \text{......(2.22.1)}$$

$$T_{\mathrm{EM}}^{01}=\frac{1}{\mu_o c}(\mathbf{E}\times\mathbf{B})_x \qquad \text{.......(2.22.2)}$$

$$T_{\mathrm{EM}}^{02}=\frac{1}{\mu_o c}(\mathbf{E}\times\mathbf{B})_y \qquad \text{........(2.22.3)}$$

$$T_{\mathrm{EM}}^{03}=\frac{1}{\mu_o c}(\mathbf{E}\times\mathbf{B})_z \qquad \text{.........(2.22.4)}$$

<u>Def 2.23</u>

Suppose at a point the E field w.r.t. time can be given by $\mathbf{E} = \mathbf{E}(t)$, then the component of this E field with frequency below ω is given by

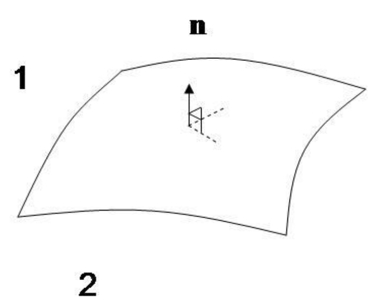
$$\mathbf{E}_\omega = \frac{1}{\sqrt{2\pi}} \int_{-\omega}^{\omega} \mathbf{F}(\omega') e^{-i\omega' t} d\omega' \qquad \text{……..(2.23.1)}$$

where $\mathbf{F}(\omega')$ is given by

$$\mathbf{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathbf{E}(t) e^{i\omega t} dt \qquad \text{……(2.23.2)}$$

Thm 2.24

It can be proved from (2.5.10) to (2.5.13) the boundary condition between two medium



$$\mathbf{n}\cdot(\mathbf{D}_1 - \mathbf{D}_2) = \sigma_f \qquad \text{……..(2.24.1a)}$$
$$\mathbf{n}\cdot(\mathbf{B}_1 - \mathbf{B}_2) = 0 \qquad \text{……..(2.24.1b)}$$
$$\mathbf{n}\times(\mathbf{E}_1 - \mathbf{E}_2) = 0 \qquad \text{……….(2.24.1c)}$$
$$\mathbf{n}\times(\mathbf{H}_1 - \mathbf{H}_2) = \boldsymbol{\kappa}_f \qquad \text{………(2.24.1d)}$$

Thm 2.25

Suppose we are in Ω and the new coordinate system described in Thm 2.16. Suppose now we change our coordinate in Ω from (ct,x,y,z) to (ct', x', y', z') given by

$$\begin{cases} x' = \gamma[x - ut] \\ t' = \gamma[t - \frac{u}{c^2} x] \\ y' = y \\ z' = z \end{cases} \quad \text{…….(2.25.1)}$$

Write $(ct, x, y, z) = (x_0, x_1, x_2, x_3)$, $(ct', x', y', z') = (x'_0, x'_1, x'_2, x'_3)$. Then the above transformation can also be written as

$$\begin{cases} x'_0 = \gamma[x_0 - \beta x_1] \\ x'_1 = \gamma[-\beta x_0 + x_1] \\ x'_2 = x_2 \\ x'_3 = x_3 \end{cases} \quad \text{…….(2.25.2)}$$

From (2.14.1), we know that our field tensor defined in Thm 2.17 is transformed as

$$F'^{\mu\nu} = \frac{\partial x'_\mu}{\partial x_\alpha} \frac{\partial x'_\nu}{\partial x_\beta} F^{\alpha\beta} \quad \text{…….(2.25.3)}$$

Substitue the definition of the field tensor in Thm 2.17, together with (2.25.2), from (2.25.3), we can prove:

$$\begin{cases} E'_x = E_x \\ E'_y = \gamma(E_y - uB_z) \\ E'_z = \gamma(E_z + uB_y) \\ B'_x = B_x \\ B'_y = \gamma(B_y + \frac{u}{c^2} E_z) \\ B'_z = \gamma(B_z - \frac{u}{c^2} E_y) \end{cases} \quad \text{……..(2.25.4a to f)}$$

It can be proved for the following two special cases:

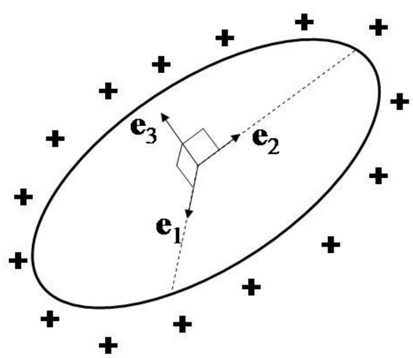
a.) $\mathbf{B}=0$ in S (orginal frame)

$$\Rightarrow \mathbf{B}' = -\frac{1}{c^2}(\mathbf{u}\times\mathbf{E}') \qquad \text{.......(2.25.5a)}$$

b.) $\mathbf{E}=0$ in S

$$\Rightarrow \mathbf{E}' = u\times\mathbf{B}' \qquad \text{.......(2.25.5b)}$$

<u>Thm 2.32</u>



Consider a circular ring such that there are +ve charges evenly distribute on it. Since it is a rigid body, we set up $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ as shown in the figure. (Meaning of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ refer to Def 1.2.) Define it external kinetic energy

$$(\text{K.E.})_{\text{ext}} = (\text{K.E.})_{\text{translational}} + \frac{1}{2}I_{11}\omega_1^2 + \frac{1}{2}I_{22}\omega_2^2 \qquad \text{......(2.32.1)}$$

and external kinetic energy

$$(\text{K.E.})_{\text{int}} = \frac{1}{2}I_{33}\omega_3^2 \qquad \text{........(2.32.2)}$$

Since the circular ring has a ω_3 rotation, it produces a magnetic moment $\boldsymbol{\mu} = \mu\mathbf{e}_3$. If we assume that ω_3 is very big such that $(\text{K.E.})_{\text{int}} >> (\text{K.E.})_{\text{ext}}$ at any time, then we can assume $(\text{K.E.})_{\text{int}}$ to be approximately the same at any time and μ in $\boldsymbol{\mu}$ keep constant.

Suppose now there is an external B field. The ring is free to move around in space. Suppose the ring is small enough such that the B field inside the ring can be consider as constant. Suppose initial the B field inside the ring is $\mathbf{B}_0$ and the ring has a magnetic moment $\boldsymbol{\mu}_0$. Also suppose finally the B field inside the ring is $\mathbf{B}_1$ and the ring has a magnetic moment $\boldsymbol{\mu}_1$. Let the initial and final external kinetic energy of the ring be denoted as $(\text{K.E.})_{\text{ext},0}$ & $(\text{K.E.})_{\text{ext},1}$. Also let the initial and final internal kinetic energy be denoted as $(\text{K.E.})_{\text{int},0}$ & $(\text{K.E.})_{\text{int},1}$. Then

$$(\text{K.E.})_{\text{int},1} = (\text{K.E.})_{\text{int},0} + (-\boldsymbol{\mu}_1 \cdot \mathbf{B}_1 + \boldsymbol{\mu}_0 \cdot \mathbf{B}_0) \qquad \ldots\ldots(2.32.3)$$

$$(\text{K.E.})_{\text{ext},1} = (\text{K.E.})_{\text{ext},0} + (-\boldsymbol{\mu}_1 \cdot \mathbf{B}_1 + \boldsymbol{\mu}_0 \cdot \mathbf{B}_0) \qquad \ldots\ldots(2.32.4)$$

Thm 2.33

In Thm 2.32, the circular ring will produce its own B field. So the B field in the space should be the sum of the external B field and the B field produce by the ring. So as the ring move freely in space, the B field must be changing. Suppose

$$W_0 = \frac{1}{2\mu_o}\int \underbrace{B}_{\text{beginning}}{}^2 d\tau \qquad \text{and} \qquad W_0 = \frac{1}{2\mu_o}\int \underbrace{B}_{\text{end}}{}^2 d\tau$$

Then we have

$$W_0 = W_1 \qquad \ldots\ldots\ldots(2.33.1)$$

That is , the energy stored in B field keep constant as the circular ring moving around in space. This can be easily understood since in all space there is no E field, and from the Poynting theorem, there need a $\mathbf{E}\cdot\mathbf{J}$ in order for the quantity $\frac{1}{2\mu_o}\int B^2 d\tau$ to be changing.

So as the circular ring move freely in space, it is a process of that internal kinetic energy transfers into external kinetic energy. This process occur because of the Lorentz force $q\dot{\mathbf{r}}_i(t)\times\mathbf{B}(\mathbf{r}_i(t))$ appear in (2.1.7).

Thm 2.34

Suppose the ring in Thm 2.32 has its e3 being fixed such that now it can only move translationally. Then $\boldsymbol{\mu}$ is fixed. Suppose the external B field is not even in space. By Lorentz force law (2.1.7), the circular ring will experience a translational force. It can be proved that this force equals to

$$\mathbf{F} = \nabla(\boldsymbol{\mu} \cdot \mathbf{B}) \qquad \text{.......(2.34.1)}$$

<u>Thm 2.37 (Helmhotz Thm)</u>

Let $\begin{cases} \nabla \cdot \mathbf{F}(\mathbf{r}) = D(\mathbf{r}) \\ \nabla \times \mathbf{F}(\mathbf{r}) = \mathbf{C}(\mathbf{r}) \end{cases}$ ....(2.37.1)

Let $D(\mathbf{r})$, $\mathbf{C}(\mathbf{r})$ both go to zero faster than $1/r^2$ as $r \to \infty$, and $\mathbf{F}(\mathbf{r})$ goes to zero as $r \to \infty$, then $\mathbf{F}(\mathbf{r})$ is uniquely given by

$$\mathbf{F}(\mathbf{r}) = -\nabla\left(\frac{1}{4\pi}\int \frac{D(\mathbf{r}')d^3r'}{|\mathbf{r}-\mathbf{r}'|}\right) + \nabla \times \left(\frac{1}{4\pi}\int \frac{\mathbf{C}(\mathbf{r}')d^3r'}{|\mathbf{r}-\mathbf{r}'|}\right)$$
$$= \frac{1}{4\pi}\int D(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r' + \frac{1}{4\pi}\int \mathbf{C}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r' \qquad \text{......(2.37.2)}$$